31AH Midterm Exam Solutions

November 13, 2018

1. Let A be a 2×2 matrix. For each of the following statements give a proof or a counterexample.

(a) If $A^2 = 0$, then $A = 0$.

(b) If $A^2 = A$, then either $A = 0$ or $A = I$.

(c) If $A^2 = I$, then either $A = I$ or $A = -I$.

Solution. Each of the statements is false.

(a) If

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

then $A^2 = 0$ but $A \neq 0$.

(b) If

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

then $A^2 = A$ but $A \neq 0$ and $A \neq I$.

(c) If

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

then $A^2 = I$ but $A \neq I$ and $A \neq -I$.

2. Let S be a square with vertices V_1, V_2, V_3, and V_4, labeled counterclockwise. Let G be the graph with vertices V_1, V_2, V_3, and V_4 and whose edges consist of the edges of S together with the diagonal connecting V_1 and V_3. Compute the number of paths in G traversing 4 edges in succession [1] that (i) start at V_1 and end at V_1, (ii) start at V_1 and end at V_2, (iii) start at V_1 and end at V_3, (iv) start at V_1 and end at V_4.

[1] such a path can traverse the same edge more than once

Solution. The answers to (i), (ii), (iii), and (iv) are given by the first column of A^4 which we know is given by $A^4\vec{e}_1$. As the adjacency matrix A of G is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Using associativity and computing recursively,

$$A\,\vec{e}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix};$$

$$A^2\,\vec{e}_1 = A\,(A\,\vec{e}_1) = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \end{bmatrix};$$

$$A^3\,\vec{e}_1 = A\,(A^2\,\vec{e}_1) = \begin{bmatrix} 4 \\ 5 \\ 5 \\ 5 \end{bmatrix};$$

$$A^4\,\vec{e}_1 = A\,(A^3\,\vec{e}_1) = \begin{bmatrix} 15 \\ 9 \\ 14 \\ 9 \end{bmatrix}.$$

Therefore, there are 15 paths of length 4 from V_1 to V_1, 9 paths of length 4 from V_1 to V_2, 14 paths of length 4 from V_1 to V_3, and 9 paths of length 4 from V_1 to V_4.

3. The following 4 points in $\mathbb{R}^3$,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix},$$

are the vertices of a parallelogram. Compute its area.

Solution. Form the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ that subtend from the first point to the other three points. Evidently,

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \vec{w} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

As $\vec{u}+\vec{v}=\vec{w}$, the parallelogram in question is spanned by the 2 vectors $\vec{u}$ and $\vec{v}$. Consequently, the area of the parallelogram is $|\vec{u}\times\vec{v}|$. We compute that

$$\vec{u}\times\vec{v}=\begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

Therefore, the area is $\sqrt{2}$.

4. Determine whether the following matrix is invertible, and if so, compute its inverse.

$$\begin{bmatrix}4&0&0&0\\0&0&2&0\\0&1&2&0\\1&0&0&1\end{bmatrix}$$

Solution. Let A denote the 4×4 matrix above. Then A is invertible if and only if the reduced row echelon form of the 4×8 matrix $[A\ I]$ has the form $[I\ B]$. Furthermore, in the case where A is invertible, $B=A^{-1}$. Therefore, we row reduce

$$\begin{bmatrix}4&0&0&0&1&0&0&0\\0&0&2&0&0&1&0&0\\0&1&2&0&0&0&1&0\\1&0&0&1&0&0&0&1\end{bmatrix}.$$

Swap rows 1 and 4 and swap rows 2 and 3:

$$\begin{bmatrix}1&0&0&1&0&0&0&1\\0&1&2&0&0&0&1&0\\0&0&2&0&0&1&0&0\\4&0&0&0&1&0&0&0\end{bmatrix}.$$

Subtract 4 times the first row from the fourth row:

$$\begin{bmatrix}1&0&0&1&0&0&0&1\\0&1&2&0&0&0&1&0\\0&0&2&0&0&1&0&0\\0&0&0&-4&1&0&0&-4\end{bmatrix}.$$

Multiply row 3 by 1/2:

$$\begin{bmatrix}1&0&0&1&0&0&0&1\\0&1&2&0&0&0&1&0\\0&0&1&0&0&1/2&0&0\\0&0&0&-4&1&0&0&-4\end{bmatrix}.$$

Subtract 2 time row 3 from row 2; also multiply row 4 by -1/4:

$$\begin{bmatrix}1&0&0&1&0&0&0&1\\0&1&0&0&0&-1&1&0\\0&0&1&0&0&1/2&0&0\\0&0&0&1&-1/4&0&0&1\end{bmatrix}.$$

Subtract row 4 from row 1:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1/4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/4 & 0 & 0 & 1 \end{bmatrix}.$$

As this last matrix is in echelon form and the first 4 columns are the 4×4 identity matrix, it follows that A is invertible and

$$A^{-1} = \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1/2 & 0 & 0 \\ -1/4 & 0 & 0 & 1 \end{bmatrix}.$$

5. Show directly from the definition that the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

are a basis for $\mathbb{R}^3$.

Solution. The vectors are a basis provided they are both linearly independent and span.

To see that the vectors are linearly independent, assume that $x_1, x_2, x_3 \in \mathbb{R}$ and

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0. \tag{1}$$

We wish to show that this assumption implies $x_1 = x_2 = x_3 = 0$. But (1) implies that

$$\begin{bmatrix} x_1 + x_2 + x_3 \\ x_2 + x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or equivalently, the system

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 + x_3 &= 0 \\ x_3 &= 0 \end{aligned}$$

obtains. As the only solution to this system is $x_1 = x_2 = x_3 = 0$, it follows that the vectors are linearly independent.

To see that the vectors span, we need to show that each vector $\vec{b} \in \mathbb{R}^3$ can be represented as a linear combination of the vectors. Accordingly, assume that

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \mathbb{R}^3.$$

We wish to show that there exist scalars $x_1, x_2, x_3 \in \mathbb{R}$ such that

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix},$$

or equivalently, the system

$$\begin{aligned} x_1 + x_2 + x_3 &= b_1 \\ x_2 + x_3 &= b_2 \\ x_3 &= b_3 \end{aligned}$$

is solvable. As $x_3 = b_3$, $x_2 = b_2 - b_3$, $x_1 = b_1 - b_2$ this system, the vectors span.